Waveguides:

solution steps:

1. find $E_x, E_y, H_x, H_y$ in terms of $E_z, H_z$
2. solve wave equation for $E_z$ (TM) or $H_z$ (TE), apply boundary conditions
3. find the other terms $E_x, E_y, H_x, H_y$ using 1
4. find phase velocity, other properties

Take out z-dependence: $\tilde{E}_x(x,y,z) = \hat{e}_x(x,y)e^{-j\beta z}$ for all terms

Solve $\nabla \times \tilde{E} = -j\omega\mu\tilde{H}$ , $\nabla \times \tilde{H} = j\omega\epsilon\tilde{E}$

$$\begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ E_x & E_y & E_z \end{vmatrix} = -j\omega\mu(\hat{x}H_x + \hat{y}H_y + \hat{z}H_z)$$

$$\frac{\partial E_z}{\partial y} + j\beta E_y = -j\omega\mu H_x \quad (x\text{-components})$$

..., and 2 other similar equations

$$\begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ H_x & H_y & H_z \end{vmatrix} = j\omega\epsilon(\hat{x}E_x + \hat{y}E_y + \hat{z}E_z)$$

$$-j\beta H_x - \frac{\partial H_z}{\partial x} = j\omega\epsilon E_y \quad (y\text{-components})$$

... and 2 other similar expressions

$$-j\omega\mu H_x = \frac{\partial E_z}{\partial y} + j\beta\frac{1}{j\omega\epsilon}\left(-j\beta H_x - \frac{\partial H_z}{\partial x}\right)$$

$$\omega^2\mu\epsilon H_x = j\omega\epsilon\frac{\partial E_z}{\partial y} + \beta^2 H_x - j\beta\frac{\partial H_z}{\partial x}$$

$k_c^2 = k^2 - \beta^2$
cutoff wavenumber

$$\boxed{H_x = \frac{1}{\omega^2\mu\epsilon - \beta^2}\left(j\omega\epsilon\frac{\partial E_z}{\partial y} - j\beta\frac{\partial H_z}{\partial x}\right)}$$

... and 3 other similar expressions

TM modes in rectangular waveguide ($E_z \neq 0$)

$$\nabla^2 \tilde{E} + k^2 \tilde{E} = 0$$

for z-component:

$$\frac{\partial^2 \tilde{E}_z}{\partial x^2} + \frac{\partial^2 \tilde{E}_z}{\partial y^2} + \underbrace{\frac{\partial^2 \tilde{E}_z}{\partial z^2}}_{= -\beta^2 \tilde{E}_z} + k^2 \tilde{E}_z = 0$$

$$\frac{\partial^2 \tilde{E}_z}{\partial x^2} + \frac{\partial^2 \tilde{E}_z}{\partial y^2} + k_c^2 \tilde{E}_z = 0$$

$k_c^2 = k^2 - \beta^2$

$$E_z(x,y) = X(x)Y(y)$$ ~ separate x and y dependence

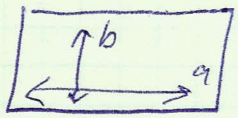
$$\frac{1}{X}\frac{\partial^2 X}{\partial x^2} + \frac{1}{Y}\frac{\partial^2 Y}{\partial y^2} + k_c^2 = 0$$

$$\frac{\partial^2 X}{\partial x^2} + X k_x^2 = 0, \quad \frac{\partial^2 Y}{\partial y^2} + k_y^2 Y = 0$$

$k_c^2 = k_x^2 + k_y^2$ separate constant into x and y parts

$$E_z = X(x)Y(y) = (\cancel{A\cos k_x x} + B\sin k_x x)(\cancel{C\cos k_y y} + D\sin k_y y)$$

$E_z = 0$ at $x=0, x=a, y=0, y=b$




$$k_x = \frac{m\pi}{a}, \quad k_y = \frac{n\pi}{b}$$

$m$ = integer, $n$ = integer

$$E_z = E_0 \sin\left(\frac{m\pi x}{a}\right)\sin\left(\frac{n\pi x}{b}\right)e^{-j\beta z}$$

from our previous derivation

$$H_x = \frac{j\omega\epsilon}{k_c^2}\left(\frac{n\pi}{b}\right)E_0 \sin\left(\frac{m\pi x}{a}\right)\cos\left(\frac{n\pi y}{b}\right)e^{-j\beta z}$$

... and 3 other similar expressions

Cutoff frequency:

$$\beta = \sqrt{k^2 - k_c^2} = \sqrt{\omega^2 \mu \epsilon - \left(\frac{m\pi}{a}\right)^2 - \left(\frac{n\pi}{b}\right)^2}$$

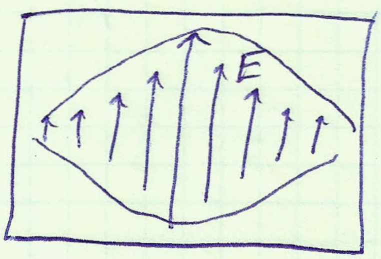
below the cutoff frequency, $\beta$ is imaginary $\rightarrow e^{-j\beta z} = e^{-\alpha z}$

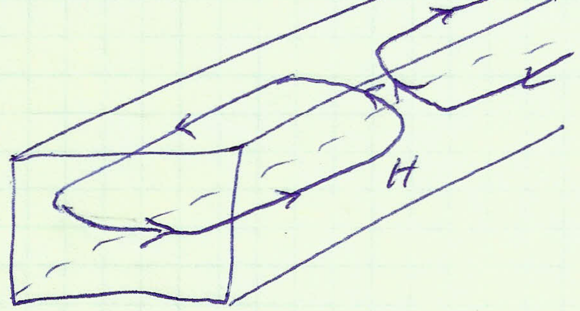
$$f_{mn} = \frac{u_{p0}}{2}\sqrt{\left(\frac{m}{a}\right)^2 + \left(\frac{n}{b}\right)^2} \qquad \beta = \frac{\omega}{u_{p0}}\sqrt{1 - \left(\frac{f_{mn}}{f}\right)^2}$$

$$u_p = \frac{\omega}{\beta} = \frac{u_{p0}}{\sqrt{1 - (f_{mn}/f)^2}} \longleftarrow \text{Both TE and TM}$$

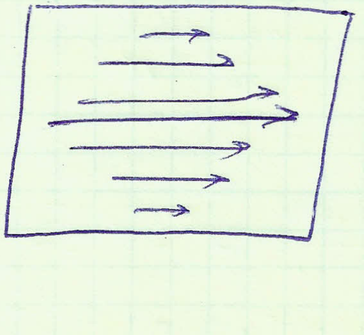
Similar analysis can be done for TE modes

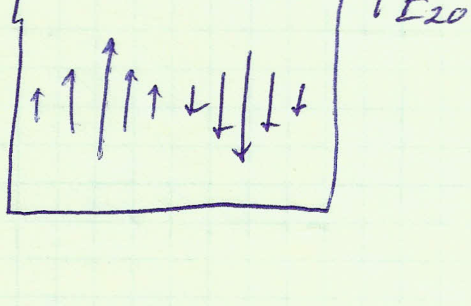
Lowest frequency mode is $TE_{10}$




other higher order modes



Propagation velocities

$u_p = \frac{\omega}{\beta}$ phase velocity

$u_g = \frac{1}{\partial\beta/\partial\omega}$ or $\frac{\partial\omega}{\partial\beta}$ group velocity

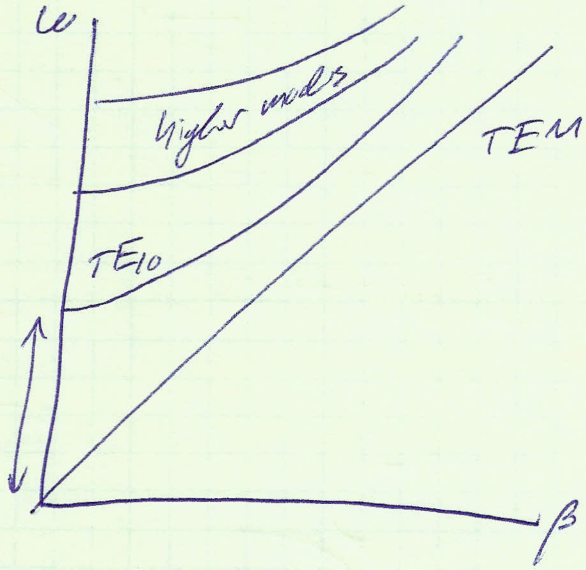
$u_g = u_{p0}\sqrt{1-(f_{mn}/f)^2}$

$u_p u_g = u_{p0}^2$

note:

$u_p > c$

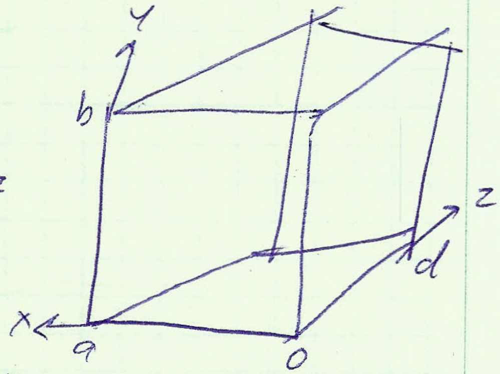
but $u_g < c$




• Resonant Cavities

start with TE mode

$$H_z = H_0 \cos\left(\frac{m\pi x}{a}\right)\cos\left(\frac{n\pi y}{b}\right)e^{-j\beta z}$$




conducting walls at $z=0$, $z=d$

$$H_z = \left(H_0^+ e^{-j\beta z} + H_0^- e^{j\beta d}\right)\cos\left(\frac{m\pi x}{a}\right)\cos\left(\frac{n\pi y}{b}\right)$$

Normal component of $H \to 0$ at conducting boundary

$$H_z = -2jH_0 \cos\left(\frac{m\pi x}{a}\right)\cos\left(\frac{n\pi y}{b}\right)\sin\left(\frac{p\pi z}{d}\right)$$

$$f_{mnp} = \frac{u_{p0}}{2}\sqrt{\left(\frac{m}{a}\right)^2 + \left(\frac{n}{b}\right)^2 + \left(\frac{p}{a}\right)^2}$$

$Q = \frac{f_{mnp}}{\Delta f}$ = energy stored / energy lost per cycle

depends on conducting and dielectric losses, coupling, etc.